

CHARACTERS OF ABELIAN GROUPS

BY

PAUL EDWIN LEWIS

B. S., Oklahoma Northeastern State Teachers College, 1931

M. S., Oklahoma Agricultural and Mechanical College, 1936

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS
IN THE GRADUATE SCHOOL OF THE
UNIVERSITY OF ILLINOIS, 1940

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Introduction

The classical definition¹ of a character of a finite abelian group A is a multiplicative mapping of the group A into the multiplicative group of all the roots of unity. If the usual definition of the product of two characters is given, the set of all characters forms an abelian group $C(A)$. One of the main results in this connection is the fact that A is isomorphic to its character group: $C(A) \approx A$.

A more comprehensive theory of the characters of abelian groups which applies to every (finite or infinite) abelian group has been constructed by Pontrjagin². In order to effect this generalization, however, Pontrjagin was forced to change the system of values of the characters from the group of roots of unity to the additive group of real numbers modulo 1.

Thus there arises quite generally the question, "Which value groups of the characters will give rise to a satisfactory theory?" In order to state the problem in a more precise form, let A and V be any two (additively written) abelian groups. As a character one may consider any additive function³ of the elements in A with values in V . If addition of characters is defined in the usual way, then the set of all the characters of A in V forms an additive abelian group $C_V(A)$.

Our problem now is to find necessary and sufficient conditions on the pair of groups A and V such that one of the following relations holds true:

- (a). $C_V(A) \approx A$
- (b). $C_V(S) \approx S$ for every subgroup S of A
- (c). $C_V(A/S) \approx A/S$ for every subgroup S of A .

The principal results concerning this problem (which are obtained as special cases of a more general theory) may be stated as follows:

¹ See e. g. E. Hecke, "Vorlesungen über die theorie der algebraischen zahlen", Leipzig (1923).

² Cf. L. Pontrjagin, "The Theory of Topological Commutative Groups", Annals of Mathematics, vol. 35 (1934), pp. 361-368.
Also see E. R. van Kampen, "Locally Bicomact Abelian Groups and Their Character Groups", Annals of Mathematics, vol. 36 (1935), pp. 448-463.

³ Cf e. g. P. Alexandroff and H. Hopf, "Topologie", Berlin (1935), p. 586.
Also H. Whitney, "Tensor products of abelian groups", Duke Mathematical Journal, vol. 4 (1938), pp. 495-528.

Theorem I: Let A be an abelian group without elements of order 0.

Then $C_V(A) \approx A$ if, and only if,

- i. A is finite.
- ii. If there is an element of order n in A , then the subgroup $_nV$ is cyclic of order n . [$_nV$ is the set of all v in V such that $nv = 0$].

Theorem II: $C_V(S) \approx S$ for every subgroup S of A if, and only if,

- i. Either A is finite or A is a direct sum of a finite number of cyclic groups of order 0.
- ii. (The same statement as ii. of Theorem I).

Theorem III: $C_V(A/S) \approx A/S$ for every subgroup S of A if, and only if,

- i. A is finite.
- ii. (The same statement as ii. of Theorem I).

Considering that the study of linear forms over a projective geometry with coordinates from a field (not necessarily commutative) shows all the essential features of such a theory of characters, it has seemed advisable to build up the whole theory immediately for abelian groups admitting a (not necessarily commutative) ring of operators. Among the restrictions imposed on the ring R one should mention that R must contain an identity element and every left-ideal must be a principal left-ideal. This restriction is necessary to insure that subgroups of cyclic groups be cyclic.

1. Properties of the Ring of Operators.

Let R be a ring (not necessarily commutative) possessing an identity element and subject to the following restrictions:

- (D) If r and s are elements in R , there exist elements m and n in R such that
 - a). $rs = mr$.
 - b). $rs = sn$.
- (E) The multiplication of left-ideals is commutative.
- (P) Left-ideals are principal left-ideals.
- (F) Every left-ideal is a product of a finite number of prime left-ideals. [A prime left-ideal in R is defined to be a greatest left-ideal different from both 0 and R].
- (U) Two products of prime left-ideals are equal if, and only if, they are the products of the same prime left-ideals apart from order of factors.

Condition (D) implies that all ideals in R are two-sided. From (D,a) and (E) it follows easily that $(st) = (s)(t)$ for every pair of ele-

ments s, t in R . Condition (P) then implies that divisors are factors, and it is a consequence of (F) and (U) that there are no zero-divisors in the ring R .

2. Cyclic Groups. The Order Ideal.

Consider an abelian group A (additively written) admitting the ring R of left-operators. The set of elements Rx formed by the totality of elements rx for r in R and x in A forms a subgroup of A . This subgroup Rx of A is called the cyclic subgroup of A generated by x : $Rx = \{x\}$. The order of an element x in A is defined to be the set $O(x)$ of elements r in R such that $rx = 0$. $O(x)$ is a left-ideal in R .

3. The Structure of Abelian Operator-groups.

A primary group A of characteristic (p) is a group in which the order-ideal of every element in A is a power of (p) . In notation: if x is any element in A , then $O(x) = (p)^{n(x)}$ for some integer $n(x) \geq 0$. The integer $n(x)$ is called the order-exponent of the element x . The set of elements x of any group G such that $O(x) = (p)^{n(x)}$ for some integer $n(x) \geq 0$ is called the primary component G_p of G of characteristic (p) . It can easily be shown that the primary components of G are subgroups of G .

The elements $x_1, x_2, \dots, x_n$ in a group A are said to be independent [over R] if the relation $\sum_{i=1}^n r_i x_i = 0$ implies that $r_i x_i = 0$ for $i=1, \dots, n$. The group A is the direct sum of the groups $A_1, A_2, \dots, A_\nu, \dots$ if

- (a) A is generated by the A_ν .
- (b) The intersection of A_ν and B_ν is 0 , where B_ν is generated by the A_μ for $\mu \neq \nu$.

If A is an abelian group without elements of order 0 , it can be proved that A is a direct sum of its primary components A_p . Also if A is an abelian group⁴ and B is a subgroup of A with the property $rB = B$ for every $r \neq 0$ in R , then B is a direct summand of A .

The primary group $A = \{x_0, x_1, \dots, x_i, \dots\}$ of characteristic (p) is said to be of type p^∞ if the relation $px_i = x_{i-1}$ holds for every integer $i > 0$ and if $O(x_0) = (p)$. If G is an abelian group with the

⁴ Cf. L. Zippin, "Countable Torsion Groups", Annals of Mathematics, vol. 36 (1935), pp. 86-99. And R. Baer, "The subgroups of the elements of finite order of an abelian group", Annals of Mathematics, vol 37 (1936), pp. 766-781. Also R. Baer, "Abelian groups that are direct summands of every containing abelian group", Bulletin of Amer. Math. Society (1940).

property that $pG = G$, then it can be shown that G is a direct sum of groups of type p^∞ . The group D is homomorphic⁵ to a group of type p^∞ if, and only if, the elements of D are not bounded.

The element x in the primary group A is of height n if the equation $p^n z = x$ has a solution z in A but the equation $p^{n+1} y = x$ does not have a solution in A . If every equation of the form $p^i y = x$ has a solution y in A , then the element x is said to have infinite height in A .

It can be shown that if A is a group generated by n elements, then every subgroup and every quotient group of A is also generated by n elements. More specifically it may be stated that every subgroup and every quotient group of A is a direct sum of a finite number of cyclic groups⁶.

4. Character Groups.

Assume that R is a ring [as in the preceding sections] satisfying conditions (D), (E), (P), (F), and (U), and consider a ring R' such that R and R' are anti-isomorphic. That is:

- (f). There is a one-to-one correspondence (denoted by $'$) between the elements r in R and r' in R' such that
- (g). $(r + s)' = r' + s'$ for every r, s in R .
- (h). $(rs)' = s'r'$ for every r, s in R .

Now suppose A and V are two abelian groups admitting the ring R as left-operators, and that in addition the group V admits the ring R' as a system of right-operators. The two operations on V are connected by

- (k). $a(vt') = (sv)t'$ for s in R , v in V , and t' in R' .

A further hypothesis imposed on the rings R, R' is the following:

- (K) 1. If v is any element in V and (r) is any ideal in R , then there exists an ideal (s') in R' satisfying:

- i. $v(s') = (r)v$
- ii. $(s) \leq (r)$.

- 2. If v is any element in V and (r') is any ideal in R' , then there exists an ideal (s) in R such that:

- i. $v(r') = (s)v$.
- ii. $(s') \leq (r')$.

It is easy to show that any subgroup of V is left-cyclic if, and

⁵ For a discussion of this theorem in the classical case see e. g. R. Baer, "Dualisms in Abelian Groups", Bulletin of the Amer. Math. Soc., (1937).

⁶ Cf. K. Asano, "Arithmetische Idealtheorie in nichtkommutativen Ringen", Japanese Journal of Mathematics, vol. 16 (1939), pp. 1-36.

only if, it is right-cyclic. Hence every left-subgroup of V is also a right-subgroup of V and conversely. The following notation will be used:

$O(x)$ = left-order of an element x in A (or V).

(r, B) = totality of elements v in B such that $rx = 0$ for r any fixed element in R .

(B, r') = totality of elements v in $B \leq V$ such that $vr' = 0$ for r' a fixed element of R' .

The mapping f of A into V is termed a character of A in V if the following three conditions are satisfied:

- 1). $f(x) = v$ where v is a uniquely determined element in V for every x in A .
- 2). $f(x + y) = f(x) + f(y)$ for every x, y in A .
- 3). $f(rx) = f(x) \cdot r'$ for r in R , and x in A .

It is clear that this definition of character coincides with the classical definition if R and R' are replaced by the ring of ordinary integers. In order that the set of characters of A in V may form an additive group also admitting the ring as left-operators, the further definitions are given:

- 4). If f and g are characters of A in V , let $(f + g)(x) = f(x) + g(x)$ for every x in A .
- 5). $(rf)(x) = r \cdot f(x)$ for r in R , x in A , and f any character of A .

If A' is a subgroup of A , f a character of A in V , and f' a character of A' in V such that $f'(x) = f(x)$ for every x in A' , then f is said to induce the character f' of A' in V . If $A = \sum_{\nu} A_{\nu}$ and if f_{ν} is a character of A_{ν} in V , then it can be shown that there exists one, and only one, character f of A in V which induces the character f_{ν} of A_{ν} in V for every ν .

If $A = A_1 + A_2 + \dots + A_n$, then it is easy to show that $C_V(A) = C_V(A_1) + C_V(A_2) + \dots + C_V(A_n)$. Also if $V = V_1 + \dots + V_m$, then $C_V(A) = C_{V_1}(A) + \dots + C_{V_n}(A)$. Another important result is the following: If A is a cyclic group ⁷ of order (r) , where (r) may be 0 or not, then $C_V(A) \approx (v, r') = (r, V)$. From this it is easy to obtain the following theorem pertaining to a sufficient condition for the isomorphism of A and its character group.

⁷ For a classical treatment of this theorem see the reference by Alexandroff and Hopf mentioned in footnote 3.

Theorem 1: If A is a direct sum of a finite number of cyclic groups:

$A = Z_1 + Z_2 + \dots + Z_n$, where $O(Z_i) = (r_i)$, and if $(V, r_i') = (r_i, V)$ is cyclic of order (r_i) , then $C_V(A) \approx A$.

With regard to necessary conditions for the isomorphism of A and its character group the following theorem is of fundamental importance.

Theorem 2: Suppose $A = \sum_v A_v$ where the A_v are cyclic of order 0 or all are of order different from 0. Then the condition $C_V(A) \approx A$ implies that only a finite number of the summands are different from 0.

5. The Main Results.

Theorem 3: Let A be an abelian group without elements of order 0. Then $C_V(A) \approx A$ if, and only if,

- a). A is a direct sum of a finite number of cyclic groups.
- b). If A contains an element of order (r), then $(V, r') = (r, V)$ is cyclic of order (r).

The sufficiency is a direct consequence of Theorem 1. For the necessity assume $C_V(A) \approx A$. Since it can be shown that A is a direct sum of cyclic groups, statement a) is a consequence of Theorem 2. Then for some integer n $A = A_{p_1} + A_{p_2} + \dots + A_{p_n}$ where $A_{p_i} \neq 0$. Therefore $C_V(A) \approx C_V(A_{p_1}) + C_V(A_{p_2}) + \dots + C_V(A_{p_n})$, and it follows that $C_V(A_{p_i}) \approx A_{p_i}$ for every primary component A_{p_i} of A. Let $A_{p_i} = Z_1 + \dots + Z_m$ where $O(Z_i) = (p^{\alpha_i})$; and let k be the maximum of the α_i . It can now be shown that V contains an element of order (p^k) . Suppose that V contains two independent elements of order (p). Then V contains a direct sum of two groups V_1 and V_2 , where each V_i contains an element of order (p). Then $C_V(A) \geq C_{V_1}(A) + C_{V_2}(A)$, and this implies there are $2m$ independent characters of order (p) in $C_V(A_{p_i})$. This is impossible since A_{p_i} contains only m independent elements of order (p) and $C_V(A_{p_i}) \approx A_{p_i}$. Hence $(p^k, V) = (V, p^{k'})$ is cyclic of order (p^k) . This statement implies b).

Theorem 4: Let A and V be abelian groups. Then $C_V(S) \approx S$ for every subgroup S of A if, and only if,

- a). A is a direct sum of a finite number of cyclic groups.
- b). If A contains an element of order (r), [which may be 0 or not], then $(r, V) = (V, r')$ is cyclic of order (r).

The sufficiency follows directly from Theorem 1. Assume $C_V(S) \approx S$ for every subgroup S of A. Then $B = \{x\}$, where x is an arbitrary element

of A , is cyclic of order say (r) . By section 4 it follows that $C_V(B) \approx (V, r') = (r, V)$, and by hypothesis $C_V(B) \approx B$; hence $B \approx (V, r')$, and V contains an element of order (r) . This is condition b).

From Theorem 1 it follows that A cannot contain both elements of order O and elements of orders different from O . If A contains no element of order O , the necessity follows from Theorem 3. Let A contain only elements (other than O) of order O . If x is any element $\neq O$ in A , then $D = \{x\}$ is cyclic of order O , and V contains an element of order O as follows from section 3. Let B be a greatest independent subset of A . Then $C_V(\{B\}) \approx \{B\}$ and Theorem 2 implies that B contains only a finite number of elements. Hence the rank of A is finite, and it can be shown that A is a direct sum of a finite number of cyclic groups of order O .

Theorem 5: Let A and V be abelian groups and suppose that the ring R is not a field. Then $C_V(A/S) \approx A/S$ for every subgroup S of A if, and only if,

- a). A is a direct sum of a finite number of cyclic groups of orders different from O .
- b). If A contains an element of order (r) , then $(r, V) = (V, r')$ is cyclic of order (r) .

The proof of the necessity is effected by showing that A cannot contain elements of order O so that the statement is a consequence of Theorem 3. Since quotient groups of cyclic groups are cyclic, the sufficiency follows from Theorem 1.

It should be mentioned that this theorem does not hold when R is a field. Its analogue in this case is

Theorem 6: Let A and V be abelian groups admitting the field R as operators. Then $C_V(A/S) \approx A/S$ for every subgroup S of A if, and only if,

- a). The rank of A is finite.
- b). V is cyclic and different from O .

The following theorem is of interest within itself.

Theorem 7: If A and V are abelian groups satisfying (E), (D), and (P), then each of the following statements implies the other:

- 1). Every character of every subgroup S of A in V is induced by a character of A in V .
- 2). If A contains an element of order $(s \cdot t)$ where s may be O or not but $t \neq O$, then $(V, s') = (s, V) \leq V \cdot t'$.

VITA

Paul Edwin Lewis was born October 6, 1911, in West Fork Arkansas. He received his education in the public schools of Oklahoma. He attended Northeastern State Teachers College of Oklahoma from 1929 to 1931, where he received the degree of Bachelor of Science in 1931. During the summer sessions from 1934 to 1936 he attended Oklahoma Agricultural and Mechanical College, where he was granted the degree of Master of Science in 1936. He taught in the public schools of Oklahoma during the years from 1931 to 1937. In June, 1938, he was married to Annie Gaie Tapp. From 1937 to 1940 he was a Graduate Assistant at the University of Illinois. He is a member of Phi Kappa Phi and the Society of Sigma Xi.

